Introduction to the

Theory of Tzitzeica Surfaces

Part 1: Preparatory Remarks

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Introduction. Display integer solutions of

$$x^2 - ny^2 = 1 \quad : \quad n \text{ not a perfect square}$$

This Diophantine problem was already more than a thousand years old (and its general solution had already been known for well more than 500 years) by the time Euler (1707–1783) mistakenly attributed William Brouncker's solution (the first by a European, in about 1660) to John Pell (1611–1685). Recently, my friend Ahmed Sebbar noticed that Pell's problem, if approached with apparatus borrowed from the theory of circulant matrices, suggests a class of generalizations which in the simplest instance[1] draws attention to an equation

$$x^2 + y^3 + z^3 - 3xyz = 1$$

that by a rotational change of variables[2] can be brought to the form

$$Z(X^2 + Y^2) = \alpha^2 \quad : \quad \alpha^2 \equiv \frac{2}{3\sqrt{3}}$$

This, if regarded as the implicit description (with respect to the Cartesian $\{X, Y, Z\}$-frame) of a surface Σ, clearly refers to a surface of revolution (about the Z-axis), of which a natural parameterization is

$$\boldsymbol{r}(u,v) = \begin{pmatrix} f(u)\cos v \\ f(u)\sin v \\ u \end{pmatrix} \quad \text{with} \quad f(u) = \alpha/\sqrt{u}$$

Sebbar remarked[3] that "Some physicists call [the surface in question] 'Jonas' hexenhut,' [but] I call it the 'Appell sphere' because it is the unit sphere for the Finsler metric $ds = \sqrt[3]{dx^3 + dy^3 + dz^3 - 3\,dxdydz}$." When I Googled

[1] See my "Simplest generalization of Pell's problem" (September 2015).
[2] See "Differential geometry of some surfaces in 3-space" (December 2015). page 21.
[3] Personal correspondence, 16 September 2015.

"hexenhut"(the term refers to the shape of the surface, though the pseudosphere, with its finite brim and sharper point, is more deserving of the name) I was referred to Chapter 3, "Tzitzeica Surfaces" in the monograph which has since that time served as my inspiration and principal source.[4]

George Tzitzeica (1873–1939), a classmate of Lebesque and student of Darboux (defended his dissertation in 1899), was a prolific Romanian mathematician[5] who early in his career created a new class of curves[6] and a new category of surfaces. That work led to development of the "affine differential geometry" of which Tzitzeica is the acknowledged founder, and from which Tzitzeica surfaces emerge as natural objects.[7]

Hans Jonas—not to be confused with the Jewish philosopher (1903–1993) of that same name—appears to have been a productive member of the Tzitzeica school of differential geometers. Rogers & Schief cite three long papers by him, dated 1915, 1921 and 1953. It is, to judge from its title,[8] in the second of those that the hexenhut—in some sense the "simplest" Tzitzeica surface—made its first appearance.

Tzitzeica's early work was then too recent to gain notice in Eisenhart's comprehensive *A Treatise on the Differential Geometry of Curves and Surfaces* (1909), but his *Transformations of Surfaces* (1923)—though it contains no mention of affine differential geometry *per se*—does cite Tzitzeica ten times, and Jonas six times; indeed, in his Preface Eisenhart mentions a respect in which his own work paralleled that of Jonas.

So much for context. My objective in these pages will be to provide an account (without reference to the alien complexities of affine differential geometry) of the most basic elements of the theory of Tzitzeica surfaces, as expounded in Rogers & Schief's Chapter 3 and other sources.

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[4] C. Rogers & W. K. Schief, *Bäcklund & Darboux Transformations: Geometry and Modern Applications in Soliton Theory* (2002).

[5] Gabriel Teodor Pripone & Rada Gogu, "Gherghe Tzitzeica–an incomplete bibliography," Balkan Journal of Geometry and its Applications **10**, 32–56 (2005: available on the web) provides a list of 106 papers published between 1898 and 1938, of which Rogers & Schief—though they devote an entire chapter to ramifications of one of his inventions—cite only three, dated 1907–1910, which come in total to only ten pages (*most* of Tzitzeica's papers run to only two of three pages).

[6] See Lewis R. Williams, "On the Tzitzeica curve equation," (2010), which is an undergraduate thesis available as a pdf file on the web.

[7] Rogers & Schief remark (page 88) that "Tzitzeica surfaces are the analogs of spheres in affine differential geometry and, indeed, are known as affine spheres or "affinsphären."

[8] "Sopra una classe di transformazioni asintotische, applicabili in particolare alle superficie la cui curvatura è proporzionale alla quatra potenza della distanza del piano tangente da un punto fisso," Ann. Mat. Pura Appl. Bologna **30**, 223–255 (1921).

Elementary mathematics relating to the asymptotic parameterization of surfaces of negative Gaussian curvature. Let $\boldsymbol{r}(u,v)$ describe (relative to a Cartesian frame) a surface Σ. From the vectors $\partial_u \boldsymbol{r} \equiv \boldsymbol{r}_u(u,v)$ and $\partial_v \boldsymbol{r} \equiv \boldsymbol{r}_v(u,v)$, which are tangent to Σ at the point $P \equiv \{u,v\}$, we construct the unit normal

$$\boldsymbol{N}(u,v) = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|}$$

and by differentiation of $\boldsymbol{N}\cdot\boldsymbol{N} = 1$ obtain $\boldsymbol{N}\cdot\boldsymbol{N}_u(u,v) = \boldsymbol{N}\cdot\boldsymbol{N}_v(u,v) = 0$, from which we learn that also $\boldsymbol{N}_u(u,v)$ and $\boldsymbol{N}_v(u,v)$ lie in that tangent plane. The 1^{st} fundamental form

$$ds^2 = d\boldsymbol{r}\cdot d\boldsymbol{r} = (\boldsymbol{r}_u du + \boldsymbol{r}_v dv)\cdot(\boldsymbol{r}_u du + \boldsymbol{r}_v dv)$$

can be written

$$ds^2 = d\boldsymbol{\xi}\cdot\mathbb{G}\,d\boldsymbol{\xi} \tag{1.1}$$

with

$$\mathbb{G}(u,v) = \begin{pmatrix} \boldsymbol{r}_u\cdot\boldsymbol{r}_u & \boldsymbol{r}_u\cdot\boldsymbol{r}_v \\ \boldsymbol{r}_u\cdot\boldsymbol{r}_v & \boldsymbol{r}_v\cdot\boldsymbol{r}_v \end{pmatrix} \equiv \begin{pmatrix} E & F \\ F & G \end{pmatrix} \quad \text{and} \quad d\boldsymbol{\xi} = \begin{pmatrix} du \\ dv \end{pmatrix} \tag{1.2}$$

The 2^{nd} fundamental form[9] $-d\boldsymbol{r}\cdot d\boldsymbol{N} = dd\boldsymbol{r}\cdot\boldsymbol{N}$ can be written

$$d\boldsymbol{\xi}\cdot\mathbb{H}\,d\boldsymbol{\xi} \tag{2.1}$$

with

$$\mathbb{H}(u,v) = \begin{pmatrix} \boldsymbol{r}_{uu}\cdot\boldsymbol{N} & \boldsymbol{r}_{uv}\cdot\boldsymbol{N} \\ \boldsymbol{r}_{uv}\cdot\boldsymbol{N} & \boldsymbol{r}_{vv}\cdot\boldsymbol{N} \end{pmatrix} \equiv \begin{pmatrix} e & f \\ f & g \end{pmatrix} \tag{2.2}$$

$\mathbb{G}$ and $\mathbb{H}$ are real symmetric matrices, so their eigenvalues are real and (when distinct) the associated eigenvectors are orthogonal. From (1) we see that $\mathbb{G}$ is in variably positive-definite: both eigenvalues are positive. No such restriction attaches, however, to $\mathbb{H}$. The Gaussian curvature at $\{u,v\}$

$$K(u,v) = \frac{\det\mathbb{H}(u,v)}{\det\mathbb{G}(u,v)} \tag{3}$$

can be therefore of either sign (or zero). Tzitzeica surfaces are surfaces of negative curvature: the eigenvalues of $\mathbb{H}$ are of opposite signs. We look in finer detail to what this says about the structure.

The eigenvalues of *any* real symmetric 2×2 matrix $\mathbb{H}$ can be described

$$\lambda_\pm = \tfrac{1}{2}\Big\{e + g \pm \sqrt{(e-g)^2 + 4f^2}\Big\} \tag{4.1}$$

$$= \tfrac{1}{2}\Big\{\operatorname{tr}\mathbb{H} \pm \sqrt{2\operatorname{tr}\mathbb{H}^2 - (\operatorname{tr}\mathbb{H})^2}\Big\} \tag{4.2}$$

$$= \tfrac{1}{2}\Big\{\operatorname{tr}\mathbb{H} \pm \sqrt{\operatorname{tr}\mathbb{H}^2 - 4\det\mathbb{H}}\Big\} \tag{4.3}$$

[9] Use $(\boldsymbol{r}_\alpha\cdot\boldsymbol{N})_\beta = (0)_\beta = \boldsymbol{r}_{\alpha\beta}\cdot\boldsymbol{N} + \boldsymbol{r}_\alpha\cdot\boldsymbol{N}_\beta$, where α,β range on $\{u,v\}$.

The reality of the eigenvalues is assured by (4.1). Equation (4.3) follows by the identity $\det\mathbb{H} = \frac{1}{2}\big\{(\mathrm{tr}\,\mathbb{H})^2 - \mathrm{tr}\,\mathbb{H}^2\big\}$ from (4.2), and immediately supplies the familiar relation $\det\mathbb{H} = \lambda_+\lambda_-$.

As the 2-vector

$$\boldsymbol{\rho}(\varphi) = \begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}$$

ranges on the unit circle the vector $\mathbb{H}\boldsymbol{\rho}(\varphi)$ ...

- In the case $\det\mathbb{H} > 0$ (eigenvalues distinct) traces a "peanut" (Figure 1), with maximal/minimal dimensions (length and waist) set by λ_+ and λ_-, orientation set by the orthonormal eigenvectors

- In the case $\det\mathbb{H} < 0$ traces a 4-lobe "radiation pattern" (Figure 2), with (possibly equal) maximal/minimal dimensions (principal lobe and side lobe) set by λ_+ and λ_-, orientation set by the orthonormal eigenvectors. There are, between the lobes, four "radiation-free" directions, identified by two vectors (and their negatives). It is those in which we have particuar interest, since it those that in the differential geometrical setting define the "asymptotic directions" from which derive the asymptotic parameterizations in terms of which the basic theory of Tzitzeica surfaces is most naturally formulated.

Write

$$\boldsymbol{z} = \begin{pmatrix}x\\ y\end{pmatrix}$$

and notice that the quadratic form $\boldsymbol{z}\cdot\mathbb{H}\,\boldsymbol{z}$ can be factored:

$$\begin{aligned}\boldsymbol{z}\cdot\mathbb{H}\,\boldsymbol{z} &= ex^2 + 2fxy + gy^2\\ &= \Big\{\sqrt{e}\,x + \frac{f+\sqrt{f^2-eg}}{\sqrt{e}}\,y\Big\}\Big\{\sqrt{e}\,x + \frac{f-\sqrt{f^2-eg}}{\sqrt{e}}\,y\Big\} &&(5.1)\\ &= \Big\{\sqrt{e}\,x + \frac{f+\sqrt{-\det\mathbb{H}}}{\sqrt{e}}\,y\Big\}\Big\{\sqrt{e}\,x + \frac{f-\sqrt{-\det\mathbb{H}}}{\sqrt{e}}\,y\Big\} &&(5.2)\end{aligned}$$

The factors are real or complex according as $\det\mathbb{H} \lesseqgtr 0$. If $\boldsymbol{z}\cdot\mathbb{H}\,\boldsymbol{z} = 0$ then one or the other of the factors must vanish. In the cases of particular interest (negative curvature, $\det\mathbb{H} < 0$) the factors are real-valued, and we are led to solutions

$$\boldsymbol{z}_1 = \begin{pmatrix}-\frac{f+\sqrt{f^2-eg}}{e}\\ \\ 1\end{pmatrix} \qquad \boldsymbol{z}_2 = \begin{pmatrix}-\frac{f-\sqrt{f^2-eg}}{e}\\ \\ 1\end{pmatrix} \tag{6}$$

that (normalized) are represented by red vectors in Figure 2, where they indicate "nodal directions" in the "radiation pattern" $\mathbb{H}\boldsymbol{\rho}(\varphi)$).

Geometric applications of the preceding material. From (5)—change $\mathbb{H}$ to $\mathbb{G}$—we see that the 1st fundamental form can be written in the factored form

$$\begin{aligned} ds^2 &= Edu^2 + 2F du dv + G dv^2 \\ &= \Big\{\sqrt{E}\,du + \frac{F+\sqrt{F^2-EG}}{\sqrt{E}}\,dv\Big\}\Big\{\sqrt{E}\,du + \frac{F-\sqrt{F^2-EG}}{\sqrt{E}}\,dv\Big\} \quad (6.1)\\ &= \Big\{\sqrt{E}\,du + \frac{F+i\sqrt{\det\mathbb{G}}}{\sqrt{E}}\,dv\Big\}\Big\{\sqrt{E}\,du + \frac{F-i\sqrt{\det\mathbb{G}}}{\sqrt{E}}\,dv\Big\} \quad (6.2)\end{aligned}$$

Equation (6) appears as equation (61) on page 92 of Eisenhart's *Treatise on the Differential Geometry of Curves and Surfaces* (1909), where it serves as the starting point for the conformal (= isothermic = isometric) parameterization of Σ. The idea there is to note that the factors (complex conjugates of one another) are inexact differential forms that can be rendered exact by the introduction suitable integrating factors.[10] This permits one to write

$$ds^2 = \lambda(p,q)\cdot(dp^2 + dq^2)$$

Of more immediate interest is the observation (compare (6)) that the 2nd fundamental form can be written

$$\begin{aligned} &e\,du^2 + 2f\,dudv + g\,dv^2 \\ &\quad= \Big\{\sqrt{e}\,du + \frac{f+\sqrt{f^2-eg}}{\sqrt{e}}\,dv\Big\}\Big\{\sqrt{e}\,du + \frac{f-\sqrt{f^2-eg}}{\sqrt{e}}\,dv\Big\} \quad (7.1)\\ &\quad= \Big\{\sqrt{e}\,du + \frac{f+\sqrt{-\det\mathbb{H}}}{\sqrt{e}}\,dv\Big\}\Big\{\sqrt{e}\,du + \frac{f-\sqrt{-\det\mathbb{H}}}{\sqrt{e}}\,dv\Big\} \quad (7.2)\end{aligned}$$

where on surfaces of negative curvature ($\det\mathbb{H} < 0$) the factors are real-valued. The differential vector

$$\boldsymbol{d} = \begin{pmatrix} du \\ dv \end{pmatrix}$$

is said to be "asymptotic" (or "self-conjugate") if and only if $\boldsymbol{d}\cdot\mathbb{H}\,\boldsymbol{d} = 0$, which

[10] See "How to construct integrating factors: applications to the isothermal parameterization of surfaces" (March, 2016). It is shown there that (for example) the pseudosphere

$$\boldsymbol{r}(u,v) = \begin{pmatrix} \operatorname{sech} u \cos v \\ \operatorname{sech} u \sin v \\ u - \tanh u \end{pmatrix} \quad : \quad ds^2 = \tanh^2 u\, du^2 + \operatorname{sech}^2 v\, dv^2$$

can be parameterized

$$\boldsymbol{r}(p,q) = \begin{pmatrix} q^{-1}\cos p \\ q^{-1}\sin p \\ \operatorname{arccosh} q - q^{-1}\sqrt{q^2-1} \end{pmatrix} \quad : \quad ds^2 = \operatorname{sech}^2 u\cdot(dp^2 + dq^2)$$

where $p(u,v) = v$, $q(u,v) = \cosh u$.

if valid at all points on a curve $u(v)$ entails the quadratic condition

$$e\,u_v^2 + 2fu_v + g = 0$$

and by (7) resolves into into two linear conditions

$$\left.\begin{aligned}\frac{du}{dv} &= -\frac{f-\sqrt{f^2-eg}}{e}\\ \frac{du}{dv} &= -\frac{f+\sqrt{f^2-eg}}{e}\end{aligned}\right\}\tag{8}$$

with the implication that *two* such curves pass through every point $\{u,v\}$ of Σ. Equations (8) are of a form $du/dv = A(u,v)$ that only in favorable cases yields to explicit analytic solution.

EXAMPLE: THE UNIT PSEUDOSPHERE [11]

From the $\boldsymbol{r}(u,v)$ described on the preceding page one computes

$$\left.\begin{aligned}E &= \tanh^2 u & e &= -\operatorname{sech} u \tanh u\\ F &= 0 & f &= 0\\ G &= \operatorname{sech}^2 u & g &= +\operatorname{sech} u \tanh u\end{aligned}\right\}\tag{9}$$

(note that $K(u,v) = \det\mathbb{H}/\det\mathbb{G} = -1$ is immediate) and (8) becomes

$$\frac{du}{dv} = \pm\sqrt{-g/e} = \pm 1 \quad\Longrightarrow\quad u(v) = u_0 \pm v \quad\Leftrightarrow\quad v(u) = v_0 \pm u$$

where u_0 and v_0 are constants of integration. The following equations serve therefore to inscribe on the pseudosphere two populations of asymptotic curves

$$\boldsymbol{r}_+(u;v_0) = \begin{pmatrix}\operatorname{sech} u\cos(v_0+u)\\ \operatorname{sech} u\sin(v_0+u)\\ u-\tanh u\end{pmatrix},\quad \boldsymbol{r}_-(u;v_0) = \begin{pmatrix}\operatorname{sech} u\cos(v_0-u)\\ \operatorname{sech} u\sin(v_0-u)\\ u-\tanh u\end{pmatrix}$$

where v_0 serves in each instance to distinguish one member of that population from all others. A unified "asymptotic parameterization of the pseudosphere" is one introduces new parameters $\{\alpha,\beta\}$ by

$$\begin{aligned}u &= \alpha+\beta\\ v &= \alpha-\beta\end{aligned}$$

Then

$$\boldsymbol{r}(u,v) = \begin{pmatrix}\operatorname{sech} u\cos v\\ \operatorname{sech} u\sin v\\ u-\tanh u\end{pmatrix} \quad\text{becomes}\quad \boldsymbol{r}(\alpha,\beta) = \begin{pmatrix}\operatorname{sech}(\alpha+\beta)\cos(\alpha-\beta)\\ \operatorname{sech}(\alpha+\beta)\sin(\alpha-\beta)\\ (\alpha+\beta)-\tanh(\alpha+\beta)\end{pmatrix}$$

which for β and α variable produces a β-parameterized family of $\boldsymbol{r}_+$ curves, and for α and β variable produces an α-parameterized family of $\boldsymbol{r}_-$ curves. I have

[11] See "Differential geometry of some surfaces in 3-space" (December, 2015), page 24.

recently demonstrated[12] that (not surprisingly) the elements of $\mathbb{G}$ transform as components of a covariant tensor of second rank, and that (more surprisingly) so also do the elements of $\mathbb{H}$, and[13] have illustrated those facts as they relate in the case $\{u, v\} \to \{\alpha, \beta\}$ to the unit pseudosphere. In that case the transformation matrix

$$\mathbb{J} = \begin{pmatrix} u_\alpha & u_\beta \\ v_\alpha & v_\beta \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

so we have

$$\begin{aligned} \mathbb{G}(\alpha,\beta) &= \mathbb{J}^{\mathsf{T}}\mathbb{G}(u,v)\,\mathbb{J} = \mathbb{J}^{\mathsf{T}} \begin{pmatrix} \tanh^2 u & 0 \\ 0 & \operatorname{sech}^2 u \end{pmatrix} \mathbb{J}\Big|_{u\to\alpha+\beta} \\ &= \begin{pmatrix} 1 & 1-2\operatorname{sech}^2(\alpha+\beta) \\ 1-2\operatorname{sech}^2(\alpha+\beta) & 1 \end{pmatrix} \end{aligned} \tag{10.1}$$

$$\begin{aligned} \mathbb{H}(\alpha,\beta) &= \mathbb{J}^{\mathsf{T}}\mathbb{H}(u,v)\,\mathbb{J} = \mathbb{J}^{\mathsf{T}} \begin{pmatrix} -\operatorname{sech} u \tanh u & 0 \\ 0 & \operatorname{sech} u \tanh u \end{pmatrix} \mathbb{J}\Big|_{u\to\alpha+\beta} \\ &= \begin{pmatrix} 0 & -2\operatorname{sech}(\alpha+\beta)\tanh(\alpha+\beta) \\ -2\operatorname{sech}(\alpha+\beta)\tanh(\alpha+\beta) & 0 \end{pmatrix} \end{aligned} \tag{10.2}$$

From the diagonal elements of $\mathbb{G}(\alpha,\beta)$ we learn that the tangent vectors $\boldsymbol{r}_\alpha(\alpha,\beta)$ and $\boldsymbol{r}_\beta(\alpha,\beta)$ are unit vectors, and therefore that

$$\boldsymbol{r}_\alpha(\alpha,\beta)\cdot\boldsymbol{r}_\beta(\alpha,\beta) = 1-2\operatorname{sech}^2(\alpha+\beta) = \cos\omega(\alpha,\beta)$$

where $\omega(\alpha,\beta)$ is the angle of incidence of the asymptotic curves that intersect at $\{\alpha,\beta\}$. So $\mathbb{G}(\alpha,\beta)$ can be written

$$\mathbb{G}(\alpha,\beta) = \begin{pmatrix} 1 & \cos\omega(\alpha,\beta) \\ \cos\omega(\alpha,\beta) & 1 \end{pmatrix} \tag{11.1}$$

from which by $K=-1$ it follows that

$$\mathbb{H}(\alpha,\beta) = \pm\begin{pmatrix} 0 & \sin\omega(\alpha,\beta) \\ \sin\omega(\alpha,\beta) & 0 \end{pmatrix} \tag{11.2}$$

and indeed, $[1-2\operatorname{sech}^2\theta]^2 + [-2\operatorname{sech}\theta\tanh\theta]^2 = 1$ as one verifies by calculation. In the notes cited above[12] I rehearse the elegant consistency argument that leads from (10) to the conclusion that $\omega(\alpha,\beta)$ is a solution of the sine-Gordon equation.

General properties of surfaces of revolution. Some of the pseudospheric results reported above—especially at (9) and (11)— are fairly distinctive. The question arises: To what extend do those results reflect *general* properties of surfaces of revolution (of which the pseudosphere is one); to what extent are they special to

[12] "Transformations of fundamental forms," (April, 2016).
[13] See pages 3–4 of the material just cited.

the pseudosphere? That is the question that motivates the following discussion.

The general surface of revolution (about the z-axis) can be parameterized

$$\boldsymbol{r}(u,v) = \begin{pmatrix} q(u)\cos v \\ q(u)\sin v \\ p(u) \end{pmatrix} \tag{12}$$

from which we compute $\boldsymbol{r}_u$, $\boldsymbol{r}_v$

$$\boldsymbol{N} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|} = \frac{1}{q\sqrt{p_u^2+q_u^2}} \begin{pmatrix} -q\,p_u\cos v \\ -q\,p_u\sin v \\ q\,q_u \end{pmatrix} \tag{13}$$

$\boldsymbol{r}_{uu}$, $\boldsymbol{r}_{uv}$, $\boldsymbol{r}_{vv}$ and by (1.2) and (2.2) obtain

$$\left.\begin{aligned} E &= p_u^2+q_u^2 & e &= (p_u^2+q_u^2)^{-\frac{1}{2}}(q_u p_{uu} - p_u q_{uu}) \\ F &= 0 & f &= 0 \\ G &= q^2 & g &= (p_u^2+q_u^2)^{-\frac{1}{2}}q\,p_u \end{aligned}\right\} \tag{14}$$

which is structurally similar to (9). The shape matrix

$$\mathbb{S} = \frac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

assumes therefore the form

$$\mathbb{S} = \frac{1}{EG}\begin{pmatrix} G & 0 \\ 0 & E \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & g \end{pmatrix} = \frac{1}{EG}\begin{pmatrix} Ge & 0 \\ 0 & Eg \end{pmatrix} = \begin{pmatrix} e/E & 0 \\ 0 & g/G \end{pmatrix} \tag{15}$$

Generally, the eigenvalues of $\mathbb{S}$ are the principal curvatures $\{k_1, k_2\}$ and $\det\mathbb{S}$ —their product—is the Gaussian curvature K. It follows therefore from (15) that for surfaces of revolution we have

$$k_1 = e/E = (p_u^2+q_u^2)^{-\frac{3}{2}}(q_u p_{uu} - p_u q_{uu}) \tag{16.1.1}$$

$$k_2 = g/G = q^{-1}(p_u^2+q_u^2)^{-\frac{1}{2}}\,p_u \tag{16.1.2}$$

$$K = eg/EG = \frac{p_u(q_u p_{uu} - p_u q_{uu})}{q\,(p_u^2+q_u^2)^2} \tag{16.2}$$

Coordinated sign ambiguities are introduced into $\{k_1, k_2\}$ by the $(p_u^2+q_u^2)^{-\frac{1}{2}}$ factors; the sign of K is that of the numerator. Equations (16.1) conform to our intuitive expectation that the principal curves on surfaces of revolution are medians and (circular) parallels, both of which are plane curves. To see this, recall that the curvature of a t-parameterized plane curve $\{x(t), y(t)\}$ is given by

$$\text{curvature} = \pm\frac{x_t y_{tt} - y_t x_{tt}}{(x_t^2+y_t^2)^{\frac{3}{2}}} \quad : \quad \text{sign fixed by convention}$$

For medians $\{p(u), q(u)\}$ we therefore have

$$k_{\text{median}} = \pm\frac{p_u q_{uu} - q_u p_{uu}}{(p_u^2+q_u^2)^{\frac{3}{2}}} = k_1 \text{ if select minus sign}$$

The parallels at u are v-parameterized circles $\{q\cos v, q\sin v\}$ of radius $q(u)$, so have curvature

$$k_{\text{parallel}} = q^{-1}(u)$$

as follows also from (16.1.2) by $q_u = 0$ (radius is constant on the u-parallel):

$$k_2 = q^{-1}\frac{p_u}{\sqrt{p_u^2}} = \pm q^{-1}(u)$$

When—as above—the theory of plane-curve curvature is applied to plane curves on surfaces the geometry of the surface forces (*via* $\mathbb{S}$) a consistent resolution of the sign ambiguities, which reduce to a single ambiguity/convention—the sign assigned to the unit normal $\boldsymbol{N}$, which controls the signs of *both* k_1 and k_2, and leaves the sign of K unchanged.

Looking now to the construction of **asymptotic curves** on surfaces of revolution, we by (8) have

$$\frac{du}{dv} = \pm\sqrt{-g/e} \tag{17}$$

Invariably E and G are positive, so from (16.2) we see that $K \gtrless 0$ according as e and g are of the same or opposite sign. Clearly, we can proceed only in the latter case; *i.e.*, when $K < 0$.[14] The solutions of (17)—since $\sqrt{-g/e}$ is a (real-valued) function of u—are obtained by functional inversion of the functions

$$v(u) = \int^u \frac{1}{\sqrt{-g/e}}du'$$

We can expect both the integration and the functional inversion to be intractable except in favorable cases, but—as will emerge—neither problem needs actually to be addressed. If the functions $u(v)$—which differ from one another only by additive constants of integration—were in hand we would have two classes of asymptotic curves inscribed on the surface or revolution Σ:[15]

$$\begin{aligned}\boldsymbol{\alpha}(v) &= \boldsymbol{r}(+u(v), v)\\ \boldsymbol{\beta}(v) &= \boldsymbol{r}(-u(v), v)\end{aligned}$$

Members of the class $\{\mathcal{C}_\alpha\}$ are distinguished from one another by the values assigned to the respective constants of integration (ditto members of the class $\{\mathcal{C}_\beta\}$). The curves $\{\mathcal{C}_\alpha\}$ and $\{\mathcal{C}_\beta\}$ serve jointly to inscribe on Σ the "asymptotic coordinate system" $\{\alpha, \beta\}$. We look to the construction of $\mathbb{G}(\alpha, \beta)$:

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[14] This is precisely the point that emerges when one compares Figure 1 with Figure 2.

[15] Since the elements of $\mathbb{G}(u, v)$, $\mathbb{H}(u, v)$ were seen at (14) to be v-independent functions of u, it might seem most natural to adopt u as the variable when describing asymptotic curves. But as I learned the hard way, the theory unfolds most simply when (as below) that role is assigned to v, which enters with characteristic simplicity into the structure of $\boldsymbol{r}(u, v)$ and speaks to the rotational invariance of surfaces of revolution.

From the asymptotic tangent vectors

$$\begin{aligned} \boldsymbol{r}_\alpha &= u_v \boldsymbol{r}_u + \boldsymbol{r}_v \\ \boldsymbol{r}_\beta &= u_v \boldsymbol{r}_u - \boldsymbol{r}_v \end{aligned} \tag{18.1}$$

we construct

$$\begin{aligned} \boldsymbol{r}_\alpha\boldsymbol{\cdot}\boldsymbol{r}_\alpha &= (u_v\boldsymbol{r}_u + \boldsymbol{r}_v)\boldsymbol{\cdot}(u_v\boldsymbol{r}_u + \boldsymbol{r}_v) = u_v u_v E + 2u_v F + G \\ \boldsymbol{r}_\alpha\boldsymbol{\cdot}\boldsymbol{r}_\beta &= (u_v\boldsymbol{r}_u + \boldsymbol{r}_v)\boldsymbol{\cdot}(u_v\boldsymbol{r}_u - \boldsymbol{r}_v) = u_v u_v E - G \\ \boldsymbol{r}_\beta\boldsymbol{\cdot}\boldsymbol{r}_\beta &= (u_v\boldsymbol{r}_u - \boldsymbol{r}_v)\boldsymbol{\cdot}(u_v\boldsymbol{r}_u - \boldsymbol{r}_v) = u_v u_v E - 2u_v F + G \end{aligned}$$

But (14) gives $F = 0$ and (17) gives $u_v u_v = -g/e$, so we have

$$\mathbb{G}(\alpha,\beta) = \begin{pmatrix} \xi E + G & \xi E - G \\ \xi E - G & \xi E + G \end{pmatrix} \tag{19.1}$$

where

$$\xi(u) = -g/e = \frac{q p_u}{p_u q_{uu} - q_u p_{uu}} \quad : \quad \text{positive real if } K < 0$$

Proceeding similarly from (18.1) and its normal counterparts

$$\begin{aligned} \boldsymbol{N}_\alpha &= u_v \boldsymbol{N}_u + \boldsymbol{N}_v \\ \boldsymbol{N}_\beta &= u_v \boldsymbol{N}_u - \boldsymbol{N}_v \end{aligned} \tag{18.2}$$

we construct

$$\begin{aligned} -\boldsymbol{r}_\alpha\boldsymbol{\cdot}\boldsymbol{N}_\alpha &= -(u_v\boldsymbol{r}_u + \boldsymbol{r}_v)\boldsymbol{\cdot}(u_v\boldsymbol{N}_u + \boldsymbol{N}_v) = u_v u_v e + 2u_u f + g \\ -\boldsymbol{r}_\alpha\boldsymbol{\cdot}\boldsymbol{N}_\beta &= -(u_v\boldsymbol{r}_u + \boldsymbol{r}_v)\boldsymbol{\cdot}(u_v\boldsymbol{N}_u - \boldsymbol{N}_v) = u_v u_v e - g \\ -\boldsymbol{r}_\beta\boldsymbol{\cdot}\boldsymbol{N}_\alpha &= -(u_v\boldsymbol{r}_u - \boldsymbol{r}_v)\boldsymbol{\cdot}(u_v\boldsymbol{N}_u + \boldsymbol{N}_v) = u_v u_v e - g \\ -\boldsymbol{r}_\beta\boldsymbol{\cdot}\boldsymbol{N}_\beta &= -(u_v\boldsymbol{r}_u - \boldsymbol{r}_v)\boldsymbol{\cdot}(u_v\boldsymbol{N}_u - \boldsymbol{N}_v) = u_v u_v e - 2u_u f + g \end{aligned}$$

which by $f = 0$ give

$$\mathbb{H}(\alpha,\beta) = \begin{pmatrix} \xi e + g & \xi e - g \\ \xi e - g & \xi e + g \end{pmatrix} = \begin{pmatrix} 0 & -2g \\ -2g & 0 \end{pmatrix} \tag{19.2}$$

The real symmetric matrix $\mathbb{G}(\alpha,\beta)$ is real and symmetric, so can by a rotation be brought to a diagonal form that exposes its eigenvalues $\{2\xi E, 2G\}$, which by a diagonal rescaling can be brought into coincidence with the eigenvalues $\{E, G\}$ of $\mathbb{G}(u,v)$. We are led thus to the transformation matrix

$$\mathbb{J} = \begin{pmatrix} \sqrt{\xi} & \sqrt{\xi} \\ 1 & -1 \end{pmatrix}$$

that achieves

$$\begin{aligned} \mathbb{J}^{\mathsf{T}}\mathbb{G}(u,v)\,\mathbb{J} &= \mathbb{G}(\alpha,\beta) \\ \mathbb{J}^{\mathsf{T}}\mathbb{H}(u,v)\,\mathbb{J} &= \mathbb{H}(\alpha,\beta) \end{aligned} \tag{20}$$

that in the pseudospheric case ($\xi = 1$) are found to reproduce (10). The elements of $\mathbb{G}(\alpha,\beta)$ and $\mathbb{H}(\alpha,\beta)$ are displayed at (19) and again at (20) as functions—inappropriately—of u. This defect can be removed when it is recalled that $\mathbb{J}$ refers to the coordinate transformation $\{u,v\} \to \{\alpha,\beta\}$. Specifically[12]

$$\mathbb{J} = \begin{pmatrix} \sqrt{\xi} & \sqrt{\xi} \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} u_\alpha & u_\beta \\ v_\alpha & v_\beta \end{pmatrix}$$

which gives

$$\begin{aligned} u &= \sqrt{\xi(u)}(\alpha + \beta) \\ v &= \alpha - \beta \end{aligned} \tag{21}$$

Let $u(w)$ denote the functional inverse of $w(u) = u/\sqrt{\xi(u)}$. Then

$$\begin{aligned} p(u) &= p(u(w)) \equiv P(w) \\ q(u) &= q(u(w)) \equiv Q(w) \end{aligned}$$

and (12) becomes

$$\boldsymbol{r}(w,v) = \begin{pmatrix} Q(w)\cos v \\ Q(w)\sin v \\ P(w) \end{pmatrix}$$

which by (21) becomes

$$\boldsymbol{r}(\alpha,\beta) = \begin{pmatrix} Q(\alpha+\beta)\cos(\alpha-\beta) \\ Q(\alpha+\beta)\sin(\alpha-\beta) \\ P(\alpha+\beta) \end{pmatrix} \tag{22}$$

In the pseudospheric case ($\xi = 1$) the u-w distinction disappears and one recovers precisely the result that appears at the bottom of page 6. As a consistency check we observe, for example, that (22) gives

$$\mathbb{G}(\alpha,\beta) = \begin{pmatrix} \boldsymbol{r}_\alpha\cdot\boldsymbol{r}_\alpha & \boldsymbol{r}_\alpha\cdot\boldsymbol{r}_\beta \\ \boldsymbol{r}_\beta\cdot\boldsymbol{r}_\alpha & \boldsymbol{r}_\beta\cdot\boldsymbol{r}_\beta \end{pmatrix} = \begin{pmatrix} (P'^2 + Q'^2) + Q^2 & (P'^2 + Q'^2) - Q^2 \\ (P'^2 + Q'^2) - Q^2 & (P'^2 + Q'^2) + Q^2 \end{pmatrix}$$

which conforms to (19.1) since

$$\begin{aligned} \xi E \pm G &= \xi\big[p_u^2(u) + q_u^2(u)\big] \pm q^2(u) \\ &= \xi\Big[\Big(\frac{1}{\sqrt{\xi}}P'\Big)^2 + \Big(\frac{1}{\sqrt{\xi}}Q'\Big)^2\Big] \pm Q^2 \end{aligned}$$

It seems to me remarkable that we have managed to achieve (21) and (22) without having actually to solve either of the intractable problems mentioned on page 9.

Let rescaled variables $\{a, b\}$ be defined by

$$\begin{aligned} \alpha &= a/\sqrt{R} \\ \beta &= b/\sqrt{R} \end{aligned} \quad \text{with} \quad R = (P'^2 + Q'^2) + Q^2 = \xi E + G$$

The associated transformation matrix

$$\mathbb{K} = \begin{pmatrix} \alpha_a & \alpha_b \\ \beta_a & \beta_b \end{pmatrix} = \frac{1}{\sqrt{R}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

sends $\mathbb{G}(\alpha,\beta)$ and $\mathbb{H}(\alpha,\beta)$ to

$$\mathbb{G}(a,b)=\mathbb{K}^{\intercal}\mathbb{G}(\alpha,\beta)\mathbb{K}=\begin{pmatrix}1 & C\\ C & 1\end{pmatrix} \quad:\quad C=\frac{\xi E-G}{\xi E+G}$$
$$\mathbb{H}(a,b)=\mathbb{K}^{\intercal}\mathbb{H}(\alpha,\beta)\mathbb{K}=\begin{pmatrix}0 & S\\ S & 0\end{pmatrix} \quad:\quad S=\frac{-2g}{\xi E+G}$$

which are structurally similar to (10). In pseudospheric cases—but not more generally—one has $C^2+S^2=1$, giving $K_{\text{pseudosphere}}=-1$.

In the special case $p(u)=u$ one has $p_u=1$ and $p_{uu}=0$, which causes many of the results reported above to assume a much simpler appearance.[16] But the resulting "natural parameterization"

$$\boldsymbol{r}(t,v)=\begin{pmatrix}r(t)\cos v\\ r(t)\sin v\\ t\end{pmatrix}$$

of the surface of revolution Σ is not always accessible, for to write $r(t)=q(u(t))$ one must be able to construct the functional inverse $u(t)$ of $p(u)=t$ which—as in the case $p(u)=u-\tanh u$ of the pseudosphere—may not be feasible.

Geodesics on the surface of revolution Σ (assume $\boldsymbol{r}(u,v)$ to have the form (12)) arise from a variational requirement that by $g_{12}=0$ leads to the Euler equation

$$\Big\{\frac{d}{du}\frac{\partial}{\partial v_u}-\frac{\partial}{\partial v}\Big\}\sqrt{g_{11}+g_{22}v_u^2}=0$$

The rotational symmetry characteristic of such surfaces is reflected in the v-independence of g_{11} and g_{22} and leads to the first integral ("conservation law" analogous to conserved angular momentum)

$$\frac{g_{22}v_u}{\sqrt{g_{11}+g_{22}v_u^2}}=c$$

from which follow a pair of first-order differential equations

$$v_u=\pm c\sqrt{\frac{g_{11}}{g_{22}(g_{22}-c^2)}}=\pm c\,\frac{\sqrt{p_u^2+q_u^2}}{q\sqrt{p_u^2-c^2}}$$

These can—in principle—be solve by quadrature, but the integral is intractable except in favorable cases, of which a few (sphere, cylindar, cone, paraboloid, hexenhut, pseudosphere) are discussed in the essays just cited.[16] The equations

$$\boldsymbol{\gamma}_\pm(u)=\boldsymbol{r}(u,v(u,\pm c))$$

serve to instribe "geodesic coordinates" on the surface Σ. Powerful insight into

[16] See "Geodesics on surfaces of revolution" (January, 2016), pages 7–10; "Clairault's theorem" (January, 2016), pages 4–12.

the form assumed by geodesics on surfaces of revolution is provided by Clairaut's theorem.[16] The construction of real-valued asymptotic curves was seen to require $K < 0$. No such restriction pertains to the construction of geodesics.

Looking to the construction of **conformal** (or "isothermal") coordinates, we saw at (14) that on surfaces of the form (12)

$$ds^2 = Edu^2 + Gdv^2 = (\sqrt{E}du + i\sqrt{G}dv)(\sqrt{E}du - i\sqrt{G}dv)$$

where $E(u) = p_u^2 + q_u^2$ and $G(u) = q^2$. The factors are differential forms (complex conjugates of one another), and from

$$\begin{aligned}\frac{\partial\sqrt{E}}{\partial v} &= 0\\ &\neq \frac{\partial\sqrt{G}}{\partial u} \quad \text{except in the cylindrical case: } q(u) = \text{constant}\end{aligned}$$

are seen to be inexact except in the special case mentioned.[17] From the first factor $\text{đ}F = (\sqrt{E}du + i\sqrt{G}dv)$ we are led to Pfaff's equation

$$\frac{dv}{du} - i\sqrt{E/G} = 0$$

we obtain

$$v(u;a) = i\int^u \sqrt{E/G}\,du' \quad : \quad a \text{ a constant of integration} \tag{23}$$

which describes an a-parameterized population of curves on the complex (u, v)-plane. Let

$$f(u, v) = a$$

provide the implicit description of the a^{th} member of that population. Then

$$\frac{d}{du}f(u,v) = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\frac{dv}{du} = \frac{da}{du} = 0$$

gives

$$\frac{\partial f}{\partial u} + i\,\frac{\partial f}{\partial v}\sqrt{E/G} = 0 \quad \text{whence} \quad i\sqrt{G}\,\frac{\partial f}{\partial u} = \sqrt{E}\,\frac{\partial f}{\partial v} \equiv i\chi\sqrt{EG}$$

With χ *thus defined* we have

$$\frac{\partial f}{\partial u} = \chi\sqrt{E} \quad \text{and} \quad \frac{\partial f}{\partial v} = i\chi\sqrt{G}$$

giving

$$\chi = \frac{1}{\sqrt{E}}\frac{\partial f}{\partial u} = \frac{1}{i\sqrt{G}}\frac{\partial f}{\partial v}$$

---

[17] Which by $q_u = 0 \Rightarrow e = 0 \Rightarrow \det\mathbb{H} = 0 \Rightarrow K = 0$ is seen to be essentially planar.

and

$$\chi \,\text{đ}F = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv = df$$

So though $\text{đ}F$ is inexact, $\chi\,\text{đ}F = df$ is exact, rendered exact by the "integrating factor" $\chi(u,v)$. Complex conjugation supplies $\bar{\chi}\,\text{đ}\bar{F} = d\bar{f}$ whence

$$ds^2 = \text{đ}F\,\text{đ}\bar{F} = \lambda(u,v)\cdot df\,d\bar{f} \quad \text{with} \quad \lambda = (\chi\bar{\chi})^{-1}$$

Resolving $f(u,v)$ into its real and imaginary parts

$$f(u,v) = x(u,v) + i\,y(u,v)$$

we arrive finally at this conformal representation

$$ds^2 = \Lambda(x,y)\cdot(dx^2 + dy^2) \tag{24}$$

of the metric structure of Σ, by means of which the geometry of Σ can be portrayed on the complex plane. Practical success of the program hinges (i) on one's ability to execute the integral (23)—which one can expect to be intractable except in favorable cases—and (ii) one one's ability to perform the functional inversions $\{x(u,v), y(u,v)\} \to \{u(x,y), v(x,y)\}$ required to construct

$$\Lambda(x,y) = \lambda(u(x,y), v(x,y))$$

and—more fundamentally—to construct

$$\boldsymbol{R}(x,y) = \boldsymbol{r}(u(x,y), v(x,y))$$

The functions $\{x(u,v), y(u,v)\}$ satisfy the Cauchy-Riemann equations

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}, \quad \frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u}$$

which are well known to be conformally invariant; there exist therefore infinitely many conformal parameterizations of any given surface Σ, whether or not it be a surface of revolution. For surfaces of the latter type the factorization of ds^2 is made particularly simple by the diagonal structure of $\mathbb{G}(u,v)$.[18] Conformal parameterizations of the sphere and pseudosphere are worked out in the source just cited. It was the latter that permitted Beltrami (1868) to devise the first explicit model of the non-Euclidean geometry contemplated by János Bolyai (1823) and Nikolai Lobachevsky (1829).[19] That work engaged the creative attention of Klein and Poincaré and the admiration of Hilbert, who considered the work initiated by Bolyai and Lobachevsky to have been one of the two greatest accomplishments of 19th century mathematics.

[18] For factorization in the general case see "How to construct integrating factors: applications to the isothermal parameterization of surfaces" (March, 2016), page 6.

[19] That story is sketched on pages 8–13 of "Pseudospheric Tales" (February, 2016).

That arbitrary surfaces Σ —not just surfaces of revolution—admit of conformal (isothermal) parameterization was established by Gauss (1822), who built upon a result special to surfaces of revolution that was obtained by Lagrange in 1779.

Definition and properties of Liouville surfaces. Surfaces of revolution, when presented in the $\{u, v\}$-parameterized form (12), can be inscribed with conformal coordinates almost trivially. For (see again page 13) the 1st fundamental form

$$ds^2 = E(u)du^2 + G(u)dv^2$$

—which informs us that the $\{u, v\}$ parameters are already orthogonal (they refer after all to medians and parallels)—can be written

$$ds^2 = G(u)\big[W(u)du^2 + dv^2\big] \quad : \quad W(u) = E(u)/G(u)$$

Introduce a new parameter w by $dw = \sqrt{W(u)}\,du$; *i.e.*, by

$$w(u) = \int^u \sqrt{W(u')}\,du' \quad : \quad \text{inversely } u = u(w)$$

Then

$$ds^2 = \lambda(w)\cdot(dw^2 + dv^2) \quad : \quad \lambda(w) = G(u(w))$$

The curves of constant w are simply medians to which we have assigned new names, (still) manifestly orthogonal to the curves of constant v.

"Liouville surfaces" are surfaces Σ that admit of parameterizations in which the 1st fundamental form assumes the structure[20]

$$ds^2 = (U(u) + V(v))\cdot(du^2 + dv^2) \tag{25}$$

Clearly, all surfaces of revolution are (set $V(v) = 0$) Liouville surfaces, but not conversely. The specialized form that Liouville assigns in (25) to the Λ-multiplier in (24) is—by no means accidentally, as will emerge—reminiscent of the additive structure of the dynamical Lagrangians

$$L = \tfrac{1}{2}u\cdot(\dot{\xi}^2 + \dot{\eta}^2) + \frac{w_1(\xi) + w_2(\eta)}{u} \quad : \quad u = u_1(\xi) + u_2(\eta)$$

that motivated Liouville to devise the ingenious separation of variables procedure that bears his name in mechanics.[21] That procedure acquires a deeper aspect when rendered in the languages of Hamiltonian mechanics and

[20] The surfaces that now bear his name were first discussed byJoseph Liouville (1809–1892) in 1846. They are not mentioned in Rogers & Schief or in Andrew Pressley's text. They make brief appearances in Manfredo do Carmo's *Differential Geometry of Curves & Surfaces* (Exercise 21, page 263) and in Barrett O'Neill's *Elementary Differential Geometry* (Exercise 18, page 339). They are treated fairly extensively in Eisenhart's *Treatise* (§§91, 93, 98).

[21] See "Kepler problem by descent from the Euler problem," (1995), §§3 & 4; Eli Snyder, "Euler's Problem: The Problem of Two Centers," (Reed College thesis, 1996).

Hamilton-Jacobi theory, and in that guise stimulated development of the general "theory of integrable systems" (of differential equations), which contributes centrally to the theory of solitons, and the flip side of which bears on the theory of chaos. Recently, the theories of *Liouville surfaces* and *Liouville separability* have (in quantum dress) fused.[22] Indeed, they are near-simultaneous inventions, fused already by Liouville himself. I turn now to an account of that pretty accomplishment.

As previously remarked, Clairault's theorem[16] (1743) provides powerful insight into the geometry of geodesics on surfaces of revolution. Does an analogous result pertain to Liouville's enlargement upon the theory of surfaces of revolution?

Liouville surfaces Σ are surfaces that admit of a parameterization such that

$$ds^2 = (U(u) + V(v)) \cdot (du^2 + dv^2)$$

Write $\{u(t), v(t)\}$ to describe a t-parameterized curve $\mathcal{C}$ on such a surface. Length on such a curve is given by[23]

$$\int ds = \int \sqrt{L}\, dt \quad : \quad L = (U(u) + V(v)) \cdot (\dot{u}^2 + \dot{v}^2) \tag{26}$$

and is extremal if $\{u(t), v(t)\}$ satisfy the coupled differential equations

$$\Big\{\frac{d}{dt}\frac{\partial}{\partial \dot{u}} - \frac{\partial}{\partial u}\Big\}\sqrt{L} = \Big\{\frac{d}{dt}\frac{\partial}{\partial \dot{v}} - \frac{\partial}{\partial v}\Big\}\sqrt{L} = 0$$

I long ago had occasion to demonstrate[24] that—while $L^p(\dot{q}, q, t)$ and $L(\dot{q}, q, t)$ generally give rise to distinct and inequivalent equations of motion—they give rise to equivalent equations of motion when

$$p(p-1)L^{p-2} \cdot \dot{L} \cdot (\partial L/\partial \dot{q}) = 0$$

and that for this to be the case it is sufficient that L be (i) t-independent and (ii) homogeneous of degree $n \neq 1$ in $\dot{q}$. The "Lagrangian" (26) is indeed t-independent and homogeneous (of degree $n = 2$), so we can drop the radical (case $p = \frac{1}{2}$) and obtain geodesic equations of the simplified (radical-free) form

$$\left.\begin{array}{l} 2\frac{d}{dt}(W\cdot \dot{u}) - W_u(\dot{u}^2 + \dot{v}^2) = 0 \\ 2\frac{d}{dt}(W\cdot \dot{v}) - W_v(\dot{u}^2 + \dot{v}^2) = 0 \end{array}\right. \quad : \quad W(u, v) = U(u) + V(v)$$

[22] See, for example, Thierry Daudé, Niky Kamran & Francois Nicoleau, "Inverse scattering at fixed energy on asymptotically hyperbolic Liouville surfaces," arXiv:1409.6229v1 [math-ph] 22 Sep 2014 and recent papers cited there.

[23] I find it convenient in this discussion to adopt notation and terminology standard to Lagrangian mechanics.

[24] See "Geometrical mechanics: Remarks commemorative of Heinrich Hertz" (1994).

which when multiplied by (respectively) $W\!\cdot\dot u$ and $W\!\cdot\dot v$ become

$$\begin{aligned}\tfrac{d}{dt}(W^2\dot u^2) - W_u\dot u W(\dot u^2+\dot v^2) &= 0\\ \tfrac{d}{dt}(W^2\dot v^2) - W_v\dot v W(\dot u^2+\dot v^2) &= 0\end{aligned}\tag{27}$$

Recall from mechanics Jacobi's observation that if

$$J(q_1,\dots,q_n,\dot q_1,\dots,\dot q_n)=\sum_{k=1}^n p_k\dot q^k - L \quad : \quad p_k\equiv \partial L/\partial\dot q^k$$

then[25]

$$\begin{aligned}\dot J &= \dot p_k\dot q^k + p_k\ddot q^k - \frac{\partial L}{\partial q^k}\dot q^k - \frac{\partial L}{\partial \dot q^k}\ddot q^k - \frac{\partial L}{\partial t}\\ &= \Big\{\dot p_k - \frac{\partial L}{\partial q^k}\Big\}\dot q^k + (p_k-p_k)\ddot q^k - \frac{\partial L}{\partial t}\\ &= 0 \quad \text{if } L \text{ is } t\text{-independent}\end{aligned}$$

In the present instance

$$\begin{aligned}J &= 2W\!\cdot(\dot u^2+\dot v^2) - W\!\cdot(\dot u^2+\dot v^2)\\ &= W\!\cdot(\dot u^2+\dot v^2) \quad \text{is seen now to be a constant, call it } E\end{aligned}$$

The geodesic equations (27) assume now the (nearly) separated form

$$\begin{aligned}\tfrac{d}{dt}(W^2\dot u^2) - \tfrac{d}{dt}U(u)E &= 0\\ \tfrac{d}{dt}(W^2\dot v^2) - \tfrac{d}{dt}V(v)E &= 0\end{aligned}$$

which give

$$\left.\begin{aligned}W^2\dot u^2 &= UE + \epsilon_1 E\\ W^2\dot v^2 &= VE + \epsilon_2 E\end{aligned}\right. \quad : \quad \epsilon_1 E \text{ and } \epsilon_1 E \text{ are constants of integration}$$

From the sum of those equations—which can be written

$$W\!\cdot W(\dot u^2+\dot v^2) = WE = WE + (\epsilon_1+\epsilon_2)E$$

—we have

$$\epsilon_1+\epsilon_2=0$$

whence

$$\begin{aligned}\dot u &= \frac{\sqrt{E(U+\epsilon)}}{U+V}\\ \dot v &= \frac{\sqrt{E(V-\epsilon)}}{U+V}\end{aligned}$$

The adjustable constant ϵ—not properly called a "separation constant" since $V(v)$ introduces v-dependence into the first and $U(u)$ introduces u-dependence into the second of those equations—serves to distinguish one geodesic from all

[25] "Jacobi's integral" follows also—and more elegantly, if not so swiftly—from Noether's theorem. When the momenta p_k are promoted to the status of independent variables J becomes the Hamiltonian, and $\dot J = 0$ becomes a statement of energy conservation.

others. We are placed now in position to write

$$\frac{du}{dv} \equiv \tan\varphi = \frac{\sqrt{U+\epsilon}}{\sqrt{V-\epsilon}}$$

and arrive thus at Liouville's analog (1846) of Clairault's theorem:

$$V(v)\sin^2\varphi - U(u)\cos^2\varphi = \epsilon \tag{28}$$

Here φ refers to the slope at v of the ϵ-geodesic.[26]

The following formula—due to Liouville[27]

$$\begin{aligned} K &= \frac{1}{\sqrt{g}}\Big[\Big(\frac{\sqrt{g}}{E}\mathit{\Gamma}^2{}_{11}\Big)_v - \Big(\frac{\sqrt{g}}{E}\mathit{\Gamma}^2{}_{12}\Big)_u\Big] \\ \mathit{\Gamma}^2{}_{11} &= g^{-1}\big\{-\tfrac{1}{2}FE_u + EF_u - \tfrac{1}{2}EG_v\big\} \\ \mathit{\Gamma}^2{}_{12} &= g^{-1}\big\{-\tfrac{1}{2}FE_v + \tfrac{1}{2}EG_u\big\} \end{aligned}$$

—permits evaluation of Gaussian curvature in terms that refer only to the metric structure of the surface (no reference to properties of the unit normal, 2[nd] fundamental form). If $\{u,v\}$ refer to a conformal parameterization of Σ then $ds^2 = \lambda(u,v)(du^2+dv^2)$ gives $E = G = \lambda(u,v), F = 0$ whence

$$\mathit{\Gamma}^2{}_{11} = -\tfrac{1}{2}\lambda^{-1}\lambda_v, \quad \mathit{\Gamma}^2{}_{11} = +\tfrac{1}{2}\lambda^{-1}\lambda_u$$

$$K = -\tfrac{1}{2}\lambda^{-1}(\partial_u^2 + \partial_v^2)\log\lambda \tag{29.1}$$

which on Liouville surfaces $\lambda = U(u) + V(v)$ becomes

$$K = \frac{U_u^2 + V_v^2 - (U+V)(U_{uu}+V_{vv})}{2(U+V)^3} \tag{29.2}$$

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[26] E. T. Whittaker (*A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (4[th] edition 1937), §43, page 67) reports that it was in 1849 that Liouville established that the equations of motion that derive from Lagrangians of the form $L = T - V$ with

$$\begin{aligned} T &= \tfrac{1}{2}\{u_1(q_1) + \cdots + u_n(q_n)\}\{v_1(q_1)\dot{q}_1^2 + \cdots + v_n(q_n)\dot{q}_n^2\} \\ V &= \frac{w_1(q_1) + \cdots + w_n(q_n)}{v_1(q_1) + \cdots + v_n(q_n)} \end{aligned}$$

are soluable by quadrature. Already by 1760 Leonard Euler had discovered the integrability of the "two centers problem," a special instance of the 3-body problem in which a mass m moves in a plane under gravitational influence of two fixed masses, M_1 and M_2. Whittaker (§53) remarks that in a suitable coordinate system this system falls within the class of systems considered by Liouville. The argument is elaborated in sources cited above.[21] Liouville surfaces are seen to fit naturally within this conceptual framework.

[27] See "Differential geometry of some surfaces in 3-space," (December 2015), pages 12-13.

In conformal coordinates the Laplace-Beltrami operator

$$\nabla^2 = \frac{1}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j$$

becomes

$$\nabla^2 = \lambda^{-1}(\partial_u^2 + \partial_v^2)$$

which on Liouville surfaces reads

$$\nabla^2 = \frac{1}{U(u)+V(v)}(\partial_u^2 + \partial_v^2)$$

Pavel Bleher, Denis Kosygin & Yakov Sinai[28] have written in elaborate detail about the eigenvalues of ∇^2 on a 2-dimensional torus with a Liouville metric, "which is in a sense the most general case of an integrable metric."

[28] "Distribution of energy levels of quantum free particle on the Liouville surface and Trace formulae," Comm. in Math. Phys. **170**, 375–403 (1995). The bibliography provides a good sense of the lively research tradition from which such work derives; quantum chaos and semiclassical approximations (not the theory of solitons) are central concerns.